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Rotating and orbiting strings in the near-horizon brane backgrounds

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Abstract: Using the Schwarzschild-type coordinates in stead of the global ones we reconstruct the classical rotating closed string solutions in the $AdS_5 \times S^5$ backgrounds. They are explicitly described by the Jacobi elliptic and trigonometrical functions of worldsheet coordinates. We study the orbiting closed string configurations in the near-horizon geometries of Dp, NS1 and NS5 branes, and derive the energy and spin of them, whose relation takes a simple form for short strings. Specially in the D5 and NS5 backgrounds we have a linear relation that the energy of the point-like string is proportional to the spin, which is associated with the spectrum of strings in the pp-wave geometries obtained by taking a special Penrose limit on the D5 and NS5 backgrounds.

Keywords: Bosonic Strings, D-branes, Penrose limit and pp-wave background.

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1. Introduction

The AdS/CFT correspondence relates the weakly coupled string theory on the $AdS_5 \times S^5$ background to the strongly coupled large N $\mathcal{N} = 4$ super Yang-Mills theory [1, 2, 3]. This conjecture has been shown in the supergravity approximation where the curvature is small and $\alpha' R$ can be neglected. However, in order to verify the correspondence in its full extent it is desirable to go beyond the supergravity approximation. Based on the observation [4, 5] that the string theory in the pp-wave background [6] is solvable, it has been shown how to identify particular string states with gauge invariant operators with large R-charge from the gauge theory side [7].

Gubser, Klebanov and Polyakov [8] have presented the semiclassical approach to string motions in $AdS_5 \times S^5$ in the global coordinates and reproduced the results of [7] by considering a particular configuration of classical rotating closed strings on S^5. They have also studied a rotating string configuration on AdS_5 stretched along the radial direction to give an energy-spin relation for this string state that can be compared with the perturbative result for conformal gauge field theories. A general string motion rotating both on AdS_5 and on S^5 in the global coordinates has been studied and a general formula relating the energy, spin and R-charge has been presented [9], which has been further generalized to give interpolating more general solutions [10]. Further interesting developments of the semiclassical approach to strings, membranes and D-branes in the $AdS \times S$-type backgrounds can be found in [11]–[17], where M- and string theories on this type of backgrounds are dual to conformal field theories on the brane worldvolumes.

The semiclassical approach is also expected to hold for more general cases such as non- or less supersymmetric cases and non-conformal cases [18]–[23]. The orbiting strings in a AdS_5 black hole background have been investigated and associated with the glueball states in the dual finite temperature $\mathcal{N}$=4 SYM theory [18]. Rotating string configurations in a confining geometry produced by an AdS charged black hole have been also studied [19], where long strings probe the interplay between confinement and finite-size effect. Further in Witten's model for QCD constructed with near-extremal Dp-branes the string configurations rotating along a circle with a fixed radius in the non-compact directions of Dp-branes, have been shown to give the dispersion relations between the energy and the linear momentum along the compact circle [19, 20]. The semiclassical approach has been applied to the membranes rotating in the G_2 holonomy backgrounds that are not $AdS \times S$-type ones and dual to $\mathcal{N}$=1 gauge theories in four dimensions and various type of energy-spin relations have been presented [21], while several folded closed string configurations have been studied in the Maldacena-Nuñez background, dual in the infra-red to $\mathcal{N}$=1 gauge theories in four dimensions [23].

Using the standard Schwarzschild-type coordinates in stead of the global ones we will reconstruct the classical solution representing a closed string rotating on AdS_5 in the $AdS_5 \times S^5$ background. We will also consider the string soliton rotating on S^5 and stretched along an angular direction in the $AdS_5 \times S^5$ background expressed by the Schwarzschild-type coordinates. This gives an insight how to study the closed string configurations rotating along an angular direction of the transverse space in the near-horizon geometries of Dp-branes which are dual to the non-conformal field theories. Further we will analyze the closed string motions in the near-horizon geometries of NS5-branes as well as NS1-branes and derive the energy-spin relations for the various backgrounds. Specially the energy-spin relations for point-like strings in the D5-brane and NS5-brane backgrounds will be associated with the string spectra in the pp-wave geometries obtained by taking a special Penrose limit on the D5-brane and NS5-brane backgrounds.

2. Rotating strings in $AdS_5 \times S^5$

We consider a rotating closed string motion in the $AdS_5 \times S^5$ background. In stead of the global metric we use the following metric in the Schwarzschild-type coordinates

$$ds^2 = \frac{r^2}{R^2}\left(-dt^2 + \sum_{i=1}^{3} dx_i^2\right) + \frac{R^2}{r^2}dr^2 + R^2\left(\sin^2\theta d\varphi^2 + d\theta^2 + \cos^2\theta d\Omega_3^2\right), \tag{2.1}$$

where $R^2 = \sqrt{\lambda}\alpha'$ with the 't Hooft coupling $\lambda = g_{YM}^2 N$ and

$$\begin{aligned} \sum_{i=1}^{3} dx_i^2 &= dl^2 + l^2\left(d\phi_1 + \sin^2\phi_1 d\phi_2^2\right), \\ d\Omega_3^2 &= d\varphi_1^2 + \cos^2\varphi_1\left(d\varphi_2^2 + \cos^2\varphi_2 d\varphi_3^2\right). \end{aligned} \tag{2.2}$$

In order to look for a classical string solution with conserved energy and angular momentum, we assume that a closed string rotates along $\phi = \phi_2$ of AdS_5 and make an ansatz for a time-dependent embedding

$$\begin{aligned} t &= t(\tau), \quad l = l(\tau,\sigma), \quad r = r(\tau,\sigma), \\ \phi &= \omega\tau, \quad \phi_1 = \theta = \frac{\pi}{2}, \ \varphi_i = 0 \qquad (i = 1,2,3) \end{aligned} \tag{2.3}$$

satisfying periodicity conditions, $l(\sigma + 2\pi) = l(\sigma)$ and $r(\sigma + 2\pi) = r(\sigma)$, where τ and σ denote the worldsheet time and space coordinates, and the angular velocity parameter ω is constant. The relevant worldsheet action for the closed string embedding of the form (2.3) is given by

$$
\begin{aligned}
I &= \int d\tau d\sigma L\,, \\
L &= -\frac{1}{4\pi\alpha'}\left[\frac{r^2}{R^2}\left(\dot{t}^2 - l^2\dot{\phi}^2 - \dot{l}^2 + l'^2\right) + \frac{R^2}{r^2}\left(-\dot{r}^2 + r'^2\right)\right],
\end{aligned} \tag{2.4}
$$

where dot and prime represent derivatives with respect to τ and σ respectively.

The variations with respect to t and ϕ lead, after one integration, to the first order differential equations

$$
r^2\dot{t} = A\,, \tag{2.5}
$$

$$
r^2 l^2\dot{\phi} = B\,, \tag{2.6}
$$

where the integration constants A, B are τ-independent. The string equations of motion for l and r read

$$
\frac{\partial}{\partial\tau}\left(r^2\dot{l}\right) - \frac{\partial}{\partial\sigma}\left(r^2 l'\right) - \dot{\phi}^2 l r^2 = 0\,, \tag{2.7}
$$

$$
R^2\left[\frac{\partial}{\partial\tau}\left(\frac{\dot{r}}{r^2}\right) - \frac{\partial}{\partial\sigma}\left(\frac{r'}{r^2}\right)\right] + \frac{R^2}{r^3}\left(\dot{r}^2 - r'^2\right) + \frac{r}{R^2}\left(\dot{t}^2 - l^2\dot{\phi}^2 - \dot{l}^2 + l'^2\right) = 0\,, \tag{2.8}
$$

while the constraints on the energy-momentum tensor of the system, which are expressed by $T_{\tau\sigma} = 0$ and $T_{\tau\tau} + T_{\sigma\sigma} = 0$, provide

$$
\frac{R^2}{r^2}\dot{r}r' + \frac{r^2}{R^2}\dot{l}l' = 0\,, \tag{2.9}
$$

$$
\frac{R^2}{r^2}\left(\dot{r}^2 + r'^2\right) + \frac{r^2}{R^2}\left(-\dot{t}^2 + \omega^2 l^2 + \dot{l}^2 + l'^2\right) = 0\,. \tag{2.10}
$$

Substituting the separable forms as $r = f(\tau)g(\sigma)$ and $l = F(\tau)G(\sigma)$ into (2.9) we have

$$
-R^4\frac{\dot{f}}{F\dot{F}f^3} = \frac{GG'g^3}{g'}\,, \tag{2.11}
$$

which should be constant to be equated with C_1. From the requirement of (2.6) we get $F = 1/f$, which is substituted into (2.11) to yield $C_1 = R^4$ together with

$$
GG' = \frac{R^4 g'}{g^3}\,. \tag{2.12}
$$

This equation can be integrated into

$$
G^2 = -\frac{R^4}{g^2} + C_2 \tag{2.13}
$$

with an integration constant C_2. Similarly the string equation (2.7) can be arranged into

$$
-\frac{\ddot{f}}{f} = \frac{1}{g^2 G}\frac{\partial}{\partial\sigma}(g^2 G') + \omega^2\,, \tag{2.14}
$$

which should be also constant to be equated with C_3. Therefore we can choose a solution $f = \cos\sqrt{C_3}\tau$ whose normalization has been absorbed into the unknown function $g(\sigma)$. The requirement of (2.5) determines the τ-dependence of t so that we have $t = \tan\sqrt{C_3}\tau$ and

$$\sqrt{C_3} = \frac{A}{g^2} \,. \tag{2.15}$$

Combining (2.13) and (2.14) we obtain the second order differential equation for G

$$G''\left(C_2 - G^2\right) + 2GG'^2 = \left(C_2 - G^2\right)\left(C_3 - \omega^2\right)G \,. \tag{2.16}$$

In the constraint (2.10) $\dot{t}$ is eliminated by (2.5) with (2.15) to give

$$\frac{g^2}{R^2}\left[-\frac{C_3}{f^2} + \left(\frac{\dot{f}G}{f}\right)^2\right] + R^2\left(\frac{\dot{f}}{f}\right)^2 = -\frac{g^2}{R^2}\left[G'^2 + \omega^2 G^2\right] - R^2\left(\frac{g'}{g}\right)^2 . \tag{2.17}$$

Through (2.13) this equation can be expressed in terms of G and f. The requirement that the τ-dependent terms should be canceled out in the left hand side of it fixes C_2 as $C_2 = 1$. The resulting σ-dependent first order differential equation becomes

$$G'^2 + (\omega^2 + C_3)G^2 - \omega^2 G^4 - C_3 = 0 \,. \tag{2.18}$$

Since differential of (2.18) with repect to σ and (2.18) itself combine to yield (2.16) with $C_2 = 1$, it suffices to solve the equation (2.18). There remains a task to manipulate the string equation (2.8), which is similarly expressed as

$$R^2\left[f\frac{\partial}{\partial\tau}\left(\frac{\dot{f}}{f^2}\right) - g\frac{\partial}{\partial\sigma}\left(\frac{g'}{g^2}\right) + \left(\frac{\dot{f}}{f}\right)^2 - \left(\frac{g'}{g}\right)^2\right] + \frac{g^2}{R^2}\left[\frac{C_3}{f^2} - \left(\frac{\dot{f}}{f}\right)^2 G^2 + G'^2 - \omega^2 G^2\right] = 0 \,. \tag{2.19}$$

Summing the τ-dependent terms including f we notice the disappearance of the τ-dependent factors so that (2.19) also reduces to (2.16). Therefore in order to solve the ordinary differential equation (2.18), we transform it through $G(\sigma) = \sqrt{C_3}G_0(u)/\omega$ with $u = \omega\sigma$ into

$$\left(\frac{dG_0}{du}\right)^2 = \left(1 - G_0^2\right)\left(1 - \frac{C_3}{\omega^2}G_0^2\right) . \tag{2.20}$$

In view of this equation the solution can be expressed in terms of Jacobi's elliptic function as

$$G_0 = \mathrm{sn}(\omega\sigma, k) \,, \qquad k = \frac{\sqrt{C_3}}{\omega} \,. \tag{2.21}$$

Here gathering together and putting $\sqrt{C_3} = \kappa$ we can express the soliton solution as

$$\begin{aligned} t &= \tan\kappa\tau \,, \qquad l = \frac{\kappa\mathrm{sn}(\omega\sigma)}{\omega\cos\kappa\tau} \,, \\ r &= \frac{R^2\cos\kappa\tau}{\mathrm{dn}(\omega\sigma)} \,. \end{aligned} \tag{2.22}$$

Since the action (2.4) is expressed as $I = \int dtd\sigma L_0,\ L_0 = \cos^2\kappa\tau L/\kappa$ where τ is regarded as a function of t, the canonical momenta conjugate to r, ϕ, l are given by

$$\Pi_r = \frac{\partial L_0}{\partial(\partial_t r)} = \frac{R^2 \dot{r}}{2\pi\alpha' r^2}\,, \quad \Pi_\phi = \frac{r^2 l^2 \dot{\phi}}{2\pi\alpha' R^2}\,, \quad \Pi_l = \frac{r^2 \dot{l}}{2\pi\alpha' R^2}\,. \tag{2.23}$$

The conserved spacetime energy of the soliton is given by

$$E = \frac{\cos^2\kappa\tau}{4\pi\alpha'\kappa}\int_0^{2\pi} d\sigma \left[\frac{r^2}{R^2}\left(\dot{t}^2 + l^2\dot{\phi}^2\right) + \frac{R^2}{r^2}\left(\dot{r}^2 + r'^2\right) + \frac{r^2}{R^2}\left(\dot{l}^2 + l'^2\right)\right], \tag{2.24}$$

which turns out, through the constraint (2.10) and the solution (2.22), to be

$$E = \frac{R^2\kappa}{2\pi\alpha'}\int_0^{2\pi} d\sigma \frac{1}{(\mathrm{dn}(\omega\sigma))^2}\,. \tag{2.25}$$

The spin of the soliton is also obtained by

$$S = \frac{R^2\omega}{2\pi\alpha'}\int_0^{2\pi} d\sigma \left(\frac{k\mathrm{sn}(\omega\sigma)}{\mathrm{dn}(\omega\sigma)}\right)^2. \tag{2.26}$$

Owing to $0 < \mathrm{dn} \le 1$ it is possible to introduce a function $\rho(\sigma)$ as $\cosh\rho(\sigma) = 1/\mathrm{dn}(\omega\sigma) = 1/\sqrt{1 - k^2\mathrm{sn}^2(\omega\sigma)}$ with $k = \kappa/\omega < 1$ and rewrite (2.25), (2.26) as

$$\begin{aligned} E &= \frac{R^2\kappa}{2\pi\alpha'}\int_0^{2\pi} d\sigma \cosh^2\rho\,, \\ S &= \frac{R^2\omega}{2\pi\alpha'}\int_0^{2\pi} d\sigma \sinh^2\rho\,. \end{aligned} \tag{2.27}$$

Thus we have recovered the expressions for the energy and spin of the soliton that were constructed in the global coordinates [9], where ρ is the radial coordinate.

The periodicity condition $\rho(\sigma + \pi) = \rho(\sigma)$ for the folded closed string that is divided into four segments, combines with the fundamental periodicity $2K$ of $\mathrm{dn}(\omega\sigma, k)$ where K is the complete elliptic integral of the first kind, to yield a relation $\omega\pi = 2K$ which indeed agrees with the periodicity condition $\kappa = (1/\sqrt{\eta})F(1/2, 1/2, 1, -1/\eta),\ 1 + \eta = 1/k^2$ presented in ref. [9]. Similarly from (2.22) r and l also satisfy the periodicity conditions, $r(\sigma + \pi) = r(\sigma)$ and $l(\sigma + 2\pi) = l(\sigma)$. The Jacobi elliptic function $\mathrm{dn}(\omega\sigma)$ starts at $\sigma = 0$ with the maximal value, $\mathrm{dn}(0) = 1$ and gradually decreases to reach the first minimum at $\sigma = K/\omega$ that is equal to $\pi/2$. Due to $\mathrm{dn}(K) = \sqrt{1 - k^2}$, the maximal radial coordinate $\rho_0 = \rho(\pi/2)$ is specified by $\cosh\rho_0 = 1/\sqrt{1 - k^2}$. Short strings correspond to $k \ll 1$, while long strings are achieved by taking k to be the ciritical value 1. The expression $\sinh\rho = k\mathrm{sn}(\omega\sigma)/\sqrt{1 - k^2\mathrm{sn}^2(\omega\sigma)}$ provides $\rho \approx (1/\sqrt{\eta})\sin\sigma$ for short strings.

Now we consider a closed string rotating along the φ-cycle of S^5 and stretched along the angular coordinate θ of S^5. We start to make an ansatz for a time-dependent embedding

$$\begin{aligned} &t = t(\tau)\,, \qquad r = r(\tau, \sigma)\,, \qquad \theta = \theta(\sigma)\,, \\ &\varphi = \nu\tau\,, \qquad x_i = \varphi_i = 0 \quad (i = 1, 2, 3) \end{aligned} \tag{2.28}$$

with a constant angular velocity parameter ν. From one energy-momentum constraint $T_{\tau\sigma} = 0$, that is $\dot{r}r'R^2/r^2 = 0$ we can take the radial coordinate to be a function of τ, $r = r(\tau)$. Then the relevant action is expressed as

$$I = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma \left[\frac{r^2}{R^2}\dot{t}^2 - R^2\sin^2\theta\dot{\varphi}^2 - \frac{R^2}{r^2}\dot{r}^2 + R^2\theta'^2\right], \tag{2.29}$$

which does not depend on t and φ so that $r^2\dot{t} = A$ should be τ-independent and $\dot{\varphi}\sin^2\theta$ is indeed τ-independent. The other constraint $T_{\tau\tau} + T_{\sigma\sigma} = 0$ is given by

$$-\frac{r^2}{R^2}\dot{t}^2 + R^2\left(\dot{\varphi}^2\sin^2\theta + \theta'^2\right) + \frac{R^2}{r^2}\dot{r}^2 = 0\,. \tag{2.30}$$

The θ equation of motion $\theta'' = -\nu^2\sin\theta\cos\theta$ has a first integral $\theta'^2 = -\nu^2\sin^2\theta + C_1$ with an integration constant C_1. On the other hand the r equation of motion is given by

$$\frac{r}{R^2}\dot{t}^2 + R^2\left[\partial_\tau\left(\frac{\dot{r}}{r^2}\right) + \frac{\dot{r}^2}{r^3}\right] = 0\,, \tag{2.31}$$

whose first integral is $\dot{r}^2 = A^2/R^4 - (r/C_2)^2$ with an integration constant C_2. The further integration yields $r = (AC_2/R^2)\cos((\tau+\tau_0)/C_2)$ with an integration constant τ_0. Substituting these solutions into (2.30) we note that the constraint is satisfied only when $C_1 = 1/C_2^2$. If we choose $C_2 = 1/\kappa, A = R^4\kappa, \tau_0 = 0$, then the soliton solution reads $t = \tan\kappa\tau, r = R^2\cos\kappa\tau$, which is compared with (2.22). The canonical momenta conjugate to r, φ are also given by $\Pi_r = \dot{r}R^2/(2\pi\alpha' r^2), \Pi_\varphi = \dot{\varphi}R^2\sin^2\theta/2\pi\alpha'$ which provide the energy and angular momentum of this system as $E = \kappa R^2/\alpha', J = (\nu R^2/2\pi\alpha')\int d\sigma\sin^2\theta$ accompanied with $\theta'^2 = -\nu^2\sin^2\theta + \kappa^2$. These expressions are the same as those presented using the global metric in ref. [8].

3. Orbiting strings in the near-horizon brane backgrounds

Using similar procedures we consider the closed string motion in the near-horizon geometry of NS5-branes whose metric is expressed in terms of the Schwarzschild-type coordinates as

$$\begin{aligned} ds^2 &= -dt^2 + \sum_{i=1}^{5} dx_i^2 + Q_5^{NS}\frac{dr^2}{r^2} + Q_5^{NS}d\Omega_3^2\,, \\ d\Omega_3^2 &= \sin^2\theta d\varphi^2 + d\theta^2 + \cos^2\theta d\psi^2 \end{aligned} \tag{3.1}$$

with the charge of NS5-brane Q_5^{NS}. The classical string configuration specified by $t = t(\tau)$, $x_i = 0$ $(i = 1, \cdots, 5)$, $r = r(\tau,\sigma)$, $\varphi = \nu\tau$, $\theta = \pi/2$, $\psi = 0$ extends along the radial r direction and rotates along the φ angle with constant angular velocity ν. The relevant action is given by

$$I = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma\left[\dot{t}^2 - Q_5^{NS}\sin^2\theta\dot{\varphi}^2 + \frac{Q_5^{NS}}{r^2}\left(-\dot{r}^2 + r'^2\right)\right], \tag{3.2}$$

where our configuration does not couple to the NS-NS B-field. Because of one conformal constraint given by $\dot{r}r'Q_5^{NS}/r^2 = 0$, r cannot have both τ and σ dependences, so that we

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choose $r = r(\sigma)$. Since the lagrangian does not depend on t and φ, it follows that $\dot{t}$ and $\dot{\varphi}$ are τ-independent. Therefore t can be parametrized by $t = \kappa\tau$ in the same way as $\varphi = \nu\tau$. The r equation of motion is given by

$$\partial_\sigma \left(\frac{r'}{r^2} \right) + \frac{r'^2}{r^3} = 0 \,, \tag{3.3}$$

that is compared with (2.31), while the other conformal constraint is written by

$$r'^2 = \frac{\kappa^2 - \nu^2 Q_5^{NS}}{Q_5^{NS}} r^2 \,, \tag{3.4}$$

whose differential with respect to σ yields (3.3). For $\kappa^2 \geq \nu^2 Q_5^{NS}$ the differential equation (3.4) has two exponential solutions $r = r_0 \exp(\pm\Omega\sigma)$ with $\Omega = ((\kappa^2 - \nu^2 Q_5^{NS})/Q_5^{NS})^{1/2}$. In ref. [24] the planetoid string solution for the spherical Rindler spacetime showed the similar exponential behaviors and a folded open string configuration was constructed. Here splitting the interval $0 \leq \sigma \leq 2\pi$ into two segments and combining the two solutions we construct a folded closed string configuration as follows:

$$r(\sigma) = \begin{cases} r_0 e^{\Omega\sigma}, \text{ for } 0 \leq \sigma \leq \pi \,, \\ r_0 e^{\Omega(2\pi - \sigma)}, \text{ for } \pi \leq \sigma \leq 2\pi \,. \end{cases} \tag{3.5}$$

The string is stretched from r_0 to $r_0 e^{\Omega\pi}$ and doubles back on itself. Thus for $r_0 > 0$ the folded closed string orbits outside the origin. The energy and angular momentum (spin) of this string configuration are obtained by $E = \kappa/\alpha'$, $J = \nu Q_5^{NS}/\alpha'$ where each segment yields the same contribution. It is convenient to define a length of the radial range as $l = r_0(e^{\Omega\pi} - 1)$.

Gathering together these expressions we derive a dispersion relation

$$E = \sqrt{\frac{Q_5^{NS}}{(\pi\alpha')^2} \ln^2 \left(\frac{l + r_0}{r_0} \right) + \frac{J^2}{Q_5^{NS}}} \,. \tag{3.6}$$

Due to the Virasoro constraint the induced metric on the worldsheet of the string is given by $ds^2 = (\kappa^2 - \nu^2 Q_5^{NS})(-d\tau^2 + d\sigma^2)$. From the string length element $dl_{phy} = \sqrt{\kappa^2 - \nu^2 Q_5^{NS}} d\sigma$ we have the physical string length $\Omega\sqrt{Q_5^{NS}}\pi$ that is compared with l. The long and short strings are specified by $\Omega \gg 1$ and $\Omega \ll 1$ respectively. The point-like string characterized by $\Omega = 0$ orbits at the radial location $r = r_0$ and has an energy-spin relation $E = J/\sqrt{Q_5^{NS}}$. Recently the same energy-spin relation has been presented in ref. [25] where further the quantum fluctuations around this point-like classical solution have been studied and shown to reproduce the spectrum of closed strings in the pp-wave background which was obtained by taking the Penrose limit along a particular class of null geodesics at a fixed radius on the NS5-brane background. The string theory in the obtained pp-wave background is dual to a subsector of LST theory.

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Let us turn to the investigation of a closed string motion in the near-horizon geometry of Dp-branes whose metric is written by

$$\begin{aligned} ds^2 &= \frac{r^{\frac{7-p}{2}}}{\sqrt{Q_p}}\left(-dt^2+\sum_{i=i}^{p}dx_i^2\right)+\frac{\sqrt{Q_p}}{r^{\frac{7-p}{2}}}\left(dr^2+r^2d\Omega_{8-p}^2\right), \\ d\Omega_{8-p}^2 &= \sin^2\theta d\varphi^2+d\theta^2+\cos^2\theta d\Omega_{6-p}^2, \\ d\Omega_{6-p}^2 &= \sum_{i=1}^{6-p}\prod_{j=1}^{i-1}\cos^2\varphi_j d\varphi_i^2 \end{aligned} \tag{3.7}$$

with the charge of Dp-brane Q_p. We will consider a closed string configuration characterized by

$$\begin{aligned} t &= \kappa\tau\,, \qquad r=r(\sigma)\,, \qquad \varphi=\nu\tau\,, \qquad \theta=\frac{\pi}{2}\,, \\ x_i &= 0 \quad (i=1,\cdots,p)\,, \qquad \varphi_i=0\ (i=1,\cdots,6-p)\,. \end{aligned} \tag{3.8}$$

The variation with repect to φ of the relevant action

$$I=-\frac{1}{4\pi\alpha'}\int d\tau d\sigma\left[\frac{r^{\frac{7-p}{2}}}{\sqrt{Q_p}}\dot{t}^2+\frac{\sqrt{Q_p}}{r^{\frac{7-p}{2}}}\left(r'^2-r^2\dot{\varphi}^2\right)\right] \tag{3.9}$$

yields $\partial_\tau(\dot{\varphi}r^{(p-3)/2})=0$, which indeed holds for $r=r(\sigma)$ with $p\neq 3$, while it is possible for r to have the τ-dependence in the special $p=3$ case as argued above. The Virasoro constraint reads

$$r'^2=\frac{\kappa^2}{Q_p}r^{7-p}-\nu^2r^2\,, \tag{3.10}$$

whose differential with respect to σ combines with (3.10) to give the r equation of motion

$$\frac{\kappa^2}{\sqrt{Q_p}}\frac{7-p}{2}r^{\frac{5-p}{2}}-\sqrt{Q_p}\left[2\frac{\partial}{\partial\sigma}\left(\frac{r'}{r^{\frac{7-p}{2}}}\right)+\frac{7-p}{2}\frac{r'^2}{r^{\frac{9-p}{2}}}+\frac{p-3}{2}\frac{\nu^2}{r^{\frac{5-p}{2}}}\right]=0\,. \tag{3.11}$$

From $r'^2\geq 0$ in (3.10) for $p\leq 4$ the radial r location of the string is restricted to $r\geq r_0\equiv(Q_p\nu^2/\kappa^2)^{1/(5-p)}$. We can use (3.10) to find r as a function of σ

$$\sigma=\pm\frac{\sqrt{Q_p}}{\kappa}\int_{r_0}^{r}\frac{dr'}{r'\sqrt{r'^{5-p}-r_0^{5-p}}}\,, \tag{3.12}$$

which could be inverted to find

$$r=\frac{r_0}{\left[\cos\left(\pm\frac{\nu(5-p)\sigma}{2}\right)\right]^{\frac{2}{5-p}}}\,. \tag{3.13}$$

Combining the two solutions in the two divided segments we construct a folded closed string orbiting outside the origin as follows:

$$r(\sigma)=\begin{cases} \dfrac{r_0}{\left[\cos\frac{\nu(5-p)}{2}\sigma\right]^{\frac{2}{5-p}}}, & \text{for } 0\leq\sigma\leq\pi\,, \\ \dfrac{r_0}{\left[\cos\frac{\nu(5-p)}{2}(2\pi-\sigma)\right]^{\frac{2}{5-p}}}, & \text{for } \pi\leq\sigma\leq 2\pi\,. \end{cases} \tag{3.14}$$

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For the D5-brane background the solution of (3.10) provides the same closed string configuration as (3.5) with $\Omega = ((\kappa^2 - \nu^2 Q_5)/Q_5)^{1/2}$ and the integration constant r_0. These solutions are substituted into the following expressions of the energy and spin on S^{8-p} of the configurations

$$\begin{aligned} E &= \frac{\kappa}{2\pi\alpha'\sqrt{Q_p}} \int_0^{2\pi} d\sigma r^{\frac{7-p}{2}} , \\ J &= \frac{\nu\sqrt{Q_p}}{2\pi\alpha'} \int_0^{2\pi} d\sigma r^{\frac{p-3}{2}} . \end{aligned} \tag{3.15}$$

The D5-brane background gives the energy and spin of closed string

$$\begin{aligned} E &= \frac{\kappa r_0}{\pi\alpha'} \frac{1}{\sqrt{Q_5}\Omega} \left(e^{\Omega\pi} - 1 \right) , \\ J &= \frac{\nu r_0}{\pi\alpha'} \frac{\sqrt{Q_5}}{\Omega} \left(e^{\Omega\pi} - 1 \right) , \end{aligned} \tag{3.16}$$

where we have also taken into account of the same contribution from the two segments of a one-fold string configuration. Although the expressions in (3.16) are different from those for the NS5-brane background they yield a similar dispersion relation

$$E = \sqrt{\left(\frac{l}{\pi\alpha'}\right)^2 + \frac{J^2}{Q_5}} , \tag{3.17}$$

which is compared with (3.6). It again reduces to a linear relation $E \simeq J/\sqrt{Q_5}$ for the point-like or short strings.

For the near-horizon geometry of D4-branes the classical closed string is characterized by the following energy and spin

$$\begin{aligned} E &= \frac{\nu}{2\kappa} J + \frac{r_0}{\pi\alpha'} \frac{\sin\frac{\nu}{2}\pi}{\cos^2\frac{\nu}{2}\pi} , \\ J &= \frac{Q_4\nu}{\kappa\pi\alpha'} \ln\left(\frac{1+\sin\frac{\nu}{2}\pi}{1-\sin\frac{\nu}{2}\pi} \right) \end{aligned} \tag{3.18}$$

with $r_0 = Q_4\nu^2/\kappa^2$, where we have performed the integral for half of string multiplied by factor 2. When $\nu \to 0$ the closed string shrinks to a point. For short strings that are characterized by $\nu \ll 1, E$ and J are expressed as $E \approx \nu J/\kappa, J \approx Q_4\nu^2/\kappa\alpha'$. They give an energy-spin relation $\sqrt{Q_4}E \approx (\alpha'/\kappa)^{1/2} J^{3/2}$. Long strings are achieved by taking ν to be the critical value 1. We parametrize the limit as $\nu = 1 - \eta, \eta \to 0$. The spin of long closed string is large as $J \approx (2Q_4/\kappa\pi\alpha') \ln(1/\pi\eta)$ and the energy is also large to be described in terms of J as

$$E \approx \frac{1}{2\kappa} J + \frac{4Q_4}{\kappa^2\pi\alpha'} e^{\frac{\kappa\pi\alpha'}{Q_4} J} . \tag{3.19}$$

Thus we note that both E and J diverge for long strings in the same way as both the energy and the spin on AdS diverge for long strings in the $AdS_5 \times S^5$ background [8, 9].

The D3-background also gives the σ-dependent solution $r = r_0/\cos(\pm\nu\sigma)$ with $r_0 = \sqrt{Q_3}\nu/\kappa$ of (3.13) besides the τ-dependent solution of (2.31), $r = R^2\cos\kappa\tau$. The σ-dependent solution is compared with the semi-infinite open string solution orbiting outside the horizon at $r = l$ in the 2+1 dimensional BTZ black hole [26] with mass $M_{BTZ} = 1$ and angular momentum $J_{BTZ} = 0$, that is expressed in terms of Jacobi's elliptic functions as $r = l\mathrm{dn}(\bar\sigma)/k\mathrm{cn}(\bar\sigma)$ with $\bar\sigma = \pm\kappa\sigma/l$ and $k^2 = (\kappa^2 - \nu^2 l^2)/\kappa^2$ in ref. [24]. The integrations in (3.15) with $p = 3$ are simply performed to give

$$E = \frac{\sqrt{Q_3}\nu}{\kappa\pi\alpha'}\tan\nu\pi\,, \qquad J = \frac{\sqrt{Q_3}\nu}{\alpha'}\,, \tag{3.20}$$

that yield a general energy-spin relation in a closed form

$$E = \frac{J}{\kappa\pi}\tan\left(\frac{\pi\alpha'}{\sqrt{Q_3}}J\right). \tag{3.21}$$

Since the induced metric on the worldsheet is described by $ds^2 = \nu^2\sqrt{Q_3}\tan^2\nu\sigma(-d\tau^2 + d\sigma^2)$ for $0 \le \sigma \le \pi$, the physical string length is obtained by $l_{phy} = Q_3^{1/4}\ln(1/\cos\nu\pi)$. For short strings, $\nu \ll 1$ it reduces to $\sqrt{Q_3}E \approx (\alpha'/\kappa)J^2$. Long strings specified by $\nu \to 1/2$ have very large energy and finite spin.

For the D2-brane background the energy and spin of the closed string configuration are evaluated as

$$\begin{aligned} E &= -\frac{\nu}{2\kappa}J + \frac{r_0}{\pi\alpha'}\frac{\sin\frac{3}{2}\nu\pi}{(\cos\frac{3}{2}\nu\pi)^{2/3}}\,, \\ J &= \frac{2\kappa r_0}{3\pi\alpha'\nu}\sqrt{2\alpha}F_1\left(\frac{1}{2}, -\frac{1}{3}, \frac{1}{2}, \frac{3}{2}; \alpha, \frac{\alpha}{2}\right), \end{aligned} \tag{3.22}$$

where $r_0^3 = Q_2\nu^2/\kappa^2, \alpha = 1 - \cos(3\nu\pi/2)$ and F_1 is Appell's hypergeometric function. From them we extract an energy-spin relation $\sqrt{Q_2}E \approx (\alpha'/\kappa)^{3/2}J^{5/2}$ for short strings, $\nu \ll 1$. For long strings specified by $\nu \to 1/3$ the energy diverges, while the spin remains finite.

The D1-brane background also gives

$$\begin{aligned} E &= -\frac{\nu}{\kappa}J + \frac{r_0}{\pi\alpha'}\frac{\sin 2\nu\pi}{\sqrt{\cos 2\nu\pi}}\,, \\ J &= \frac{\sqrt{Q_1}}{\pi\alpha' r_0}\left[\sqrt{2}E(\alpha, \frac{1}{\sqrt{2}}) - \frac{1}{\sqrt{2}}F(\alpha, \frac{1}{\sqrt{2}})\right] \end{aligned} \tag{3.23}$$

with $r_0^4 = Q_1\nu^2/\kappa^2$, where F and E are the elliptic integral of the first and second kinds respectively, and $\sin\alpha = \sqrt{2}\sin\nu\pi$. Short strings have a similar energy-spin relation $\sqrt{Q_1}E \approx (\alpha'/\kappa)^2 J^3$. Long strings obtained by $\nu \to 1/4$ have very large energy with finite spin.

Gathering together we observe that the energy-spin relations for short orbiting strings in the near-horizon geometries of Dp-branes ($p = 1, \cdots, 5$) can be summarized in a compact form as

$$\sqrt{Q_p}E \approx \left(\frac{\alpha'}{\kappa}\right)^{\frac{5-p}{2}} J^{\frac{7-p}{2}}\,, \tag{3.24}$$

where $p = 5$ is the special case such that the energy is linearly related to the spin. The intriguing linear behavior $\sqrt{Q_5}E \approx J$ for the point-like string orbiting at the radial location $r = r_0$ in the D5-brane background can be directly compared with the spectrum of closed strings in the geometry produced by taking the Penrose limit along a special null geodesic which stays at a fixed radius. This kind of special Penrose limit yields a solvable string theory and we write down here the string spectrum presented in ref. [27]

$$\sqrt{s}E - J = \sum_n \left[e^{U_0} \frac{s}{J} N_n^{r,\omega} |n| + \sqrt{1 + e^{2U_0} \frac{s^2}{J^2} n^2 N_n^y} \right], \tag{3.25}$$

where s is the number of D5-branes and U_0 specifies the fixed radial coordinate, and $N_n^{r,\omega}$ as well as N_n^y are the oscillation occupation numbers of massless and massive scalars respectively. Thus the ground state satisfying the classical relation $\sqrt{s}E = J$ corresponds to the classical point-like string configuration.

Moreover, we consider a closed string motion in the near-horizon geometry of NS1-branes with metric

$$ds^2 = \frac{r^6}{Q_1^{NS}} \left(-dt^2 + dx^2\right) + dr^2 + r^2 \left(\sin^2\theta d\varphi^2 + d\theta^2 + \cos^2\theta d\Omega_5^2\right), \tag{3.26}$$

where $d\Omega_5^2$ is described in terms of φ_i $(i = 1, \cdots, 5)$ by an expression similar to $d\Omega_{6-p}^2$ in (3.7) and Q_1^{NS} is the charge of NS1-brane. For the closed string configuration similarly expressed by (3.8) with $p = 1$, which does not couple to the NS-NS B-field, we see that the radial equation of motion essentially becomes the same as that for the D1-brane background, although the two metrics are different. The energy and spin of this configuration are given by

$$\begin{aligned} E &= \frac{\nu}{2\kappa} J + \frac{\sqrt{Q_1^{NS}}\nu^2}{4\kappa^2\pi\alpha'} \frac{\sin 2\nu\pi}{\cos^2 2\nu\pi}, \\ J &= \frac{\sqrt{Q_1^{NS}}\nu}{4\kappa\pi\alpha'} \ln \frac{1 + \sin 2\nu\pi}{1 - \sin 2\nu\pi}, \end{aligned} \tag{3.27}$$

which have structures similar to (3.18) for the D4-brane background. For short strings they give an energy-spin relation $(Q_1^{NS})^{1/4} E \approx (\alpha'/\kappa)^{1/2} J^{3/2}$. Long string are produced by taking a limit $\nu \to 1/4$ and have an energy-spin relation

$$E \approx \frac{1}{8\kappa} J + \frac{\sqrt{Q_1^{NS}}}{(16\kappa)^2 \pi\alpha'} e^{\frac{16\kappa\pi\alpha'}{\sqrt{Q_1^{NS}}} J}, \tag{3.28}$$

which shows the same behavior as (3.19).

4. Conclusion

Manipulating the Schwarzschild-type coordinates for the $AdS_5 \times S^5$ background, we have analyzed the motion of the closed string rotating on AdS_5 and stretched simultaneously on the radial coordinate l in the non-compact directions of the D3-branes and the radial coordinate r in the transversal directions. We have demonstrated that such two radial coordinates for the soliton solution are characterized by products of the trigonometrical function of the worldsheet time τ and Jacobi's elliptic function of the worldsheet coordinate σ, while in the global coordinates the corresponding radial coordinate ρ is explicitly described by Jacobi's elliptic function of σ. This product form is a new type of behavior which is not seen in the planetoid string solutions in various curved spacetimes [28, 29, 24]. In the Schwarzschild-type coordinates we have also reconstructed a classical solution of the closed string rotating on S^5 and stretched along an angular direction, whose radial coordinate r is specified by the trigonometrical function of τ, which corresponds to the rotating point-like string staying at the origin of the radial coordinate ρ in the global coordinates.

Based on the similar prescription we have studied the orbiting closed strings on the near-horizon geometries of Dp-branes ($1 \leq p \leq 5$), NS1-branes and NS5-branes, which are expressed in terms of the Schwarzschild-type coordinates, and shown that their radial behaviors are characterized by the trigonometrical function of σ for Dp-branes ($1 \leq p \leq 4$) and NS1-branes, and the exponential function of σ for D5-branes and NS5-branes. We have noted that the D3-brane background is so special that there are two type of closed string solutions, a τ-dependent configuration for the radial coordinate and a σ-dependent one. There is some resemblance between the energy-spin relation for the D4-brane background and that for the NS1-brane background. Although the classical energies of closed strings are implicitly given by the complicated functions of the spins, we have extracted a simple general formula of energy-spin relation for short strings in the Dp-brane backgrounds. The energies of short strings show power behaviors of the spins and specially the D5-brane as well as NS5-brane backgrounds yield a linear behavior for point-like strings. We have observed that this linear relation coincides with the linear one for the ground state energy of the closed string spectrum in the pp-wave backgrounds produced by taking a special Penrose limit along the null geodesics such that the radial coordinate is constant on the near-horizon geometries of the D5-branes and NS5-branes. It is compared with the linear relation seen in the energy and angular momentum of the point-like string sitting at $\rho = 0$ and rotating on S^5 for the $AdS_5 \times S^5$ background in the global coordinates [8].

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[4] R.R. Metsaev, *Type-IIB Green-Schwarz superstring in plane wave Ramond-Ramond background*, *Nucl. Phys.* **B 625** (2002) 70 [hep-th/0112044].

[5] R.R. Metsaev and A.A. Tseytlin, *Exactly solvable model of superstring in plane wave Ramond-Ramond background*, *Phys. Rev.* **D 65** (2002) 126004 [hep-th/0202109].

[6] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *Penrose limits and maximal supersymmetry*, *Class. and Quant. Grav.* **19** (2002) L87 [hep-th/0201081]; *A new maximally supersymmetric background of iib superstring theory*, *J. High Energy Phys.* **01** (2002) 047 [hep-th/0110242].

[7] D. Berenstein, J.M. Maldacena and H. Nastase, *Strings in flat space and PP waves from $N = 4$ super Yang-Mills*, *J. High Energy Phys.* **04** (2002) 013 [hep-th/0202021].

[8] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *A semi-classical limit of the gauge/string correspondence*, *Nucl. Phys.* **B 636** (2002) 99 [hep-th/0204051].

[9] S. Frolov and A.A. Tseytlin, *Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$*, *J. High Energy Phys.* **06** (2002) 007 [hep-th/0204226].

[10] J.G. Russo, *Anomalous dimensions in gauge theories from rotating strings in $AdS_5 \times S^5$*, *J. High Energy Phys.* **06** (2002) 038 [hep-th/0205244].

[11] G. Mandal, N.V. Suryanarayana and S.R. Wadia, *Aspects of semiclassical strings in AdS_5*, *Phys. Lett.* **B 543** (2002) 81 [hep-th/0206103].

[12] M. Alishahiha and M. Ghasemkhani, *Orbiting membranes in M-theory on $AdS_7 \times S^4$ background*, *J. High Energy Phys.* **08** (2002) 046 [hep-th/0206237].

[13] J.A. Minahan, *Circular semiclassical string solutions on $AdS_5 \times S^5$*, *Nucl. Phys.* **B 648** (2003) 203 [hep-th/0209047].

[14] A.A. Tseytlin, *Semiclassical quantization of superstrings: $AdS_5 \times S^5$ and beyond*, *Int. J. Mod. Phys.* **A 18** (2003) 981 [hep-th/0209116].

[15] R.C. Rashkov and K.S. Viswanathan, *Rotating strings with b-field*, hep-th/0211197.

[16] A. Loewy and Y. Oz, *Large spin strings in AdS_3*, *Phys. Lett.* **B 557** (2003) 253 [hep-th/0212147].

[17] P. Ouyang, *Semiclassical quantization of giant gravitons*, hep-th/0212228.

[18] A. Armoni, J.L.F. Barbon and A.C. Petkou, *Orbiting strings in AdS black holes and $N = 4$ SYM at finite temperature*, *J. High Energy Phys.* **06** (2002) 058 [hep-th/0205280].

[19] A. Armoni, J.L.F. Barbon and A.C. Petkou, *Rotating strings in confining AdS/CFT backgrounds*, *J. High Energy Phys.* **10** (2002) 069 [hep-th/0209224].

[20] M. Alishahiha and A.E. Mosaffa, *Circular semiclassical string solutions on confining AdS/CFT backgrounds*, *J. High Energy Phys.* **10** (2002) 060 [hep-th/0210122].

[21] S.A. Hartnoll and C. Núñez, *Rotating membranes on G_2 manifolds, logarithmic anomalous dimensions and $\mathcal{N} = 1$ duality*, *J. High Energy Phys.* **02** (2003) 049 [hep-th/0210218].

[22] E.G. Gimon, L.A. Pando Zayas, J. Sonnenschein and M.J. Strassler, *A soluble string theory of hadrons*, hep-th/0212061.

[23] J.M. Pons and P. Talavera, *Semi-classical string solutions for $N = 1$ SYM*, hep-th/0301178.

[24] S. Kar and S. Mahapatra, *Planetoid strings: solutions and perturbations*, *Class. and Quant. Grav.* **15** (1998) 1421 [hep-th/9701173].

[25] M. Alishahiha and A.E. Mosaffa, *Semiclassical string solutions on deformed NS5-brane backgrounds and new PP-wave*, hep-th/0302005.

[26] M. Bañados, C. Teitelboim and J. Zanelli, *The black hole in three-dimensional space-time*, *Phys. Rev. Lett.* **69** (1992) 1849 [hep-th/9204099].

[27] Y. Oz and T. Sakai, *Penrose limit and six-dimensional gauge theories*, *Phys. Lett.* **B 544** (2002) 321 [hep-th/0207223].

[28] H.J. de Vega, A.L. Larsen and N. Sanchez, *Semiclassical quantization of circular strings in de Sitter and anti-de Sitter space-times*, *Phys. Rev.* **D 51** (1995) 6917 [hep-th/9410219].

[29] H.J. de Vega and I.L. Egusquiza, *Planetoid string solutions in* $3+1$ *axisymmetric spacetimes*, *Phys. Rev.* **D 54** (1996) 7513 [hep-th/9607056].